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Necessary and sufficient conditions for the quantum Zeno and anti-Zeno effect

Harald Atmanspacher[1,2], **Werner Ehm**[2] **and Tilmann Gneiting**[3]

[1] Max-Planck-Institut für extraterrestrische Physik, 85740 Garching, Germany
[2] Institut für Grenzgebiete der Psychologie, Wilhelmstr 3a, 79098 Freiburg, Germany
[3] Department of Statistics, University of Washington, Box 354322, Seattle, WA 98195-4322, USA

E-mail: ehm@igpp.de

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Abstract

A necessary and sufficient condition for the occurrence of the quantum Zeno effect is given, refining a recent conjecture of Luo, Wang and Zhang. An analogous condition is derived for the quantum anti-Zeno effect. Both results rely on a formal connection between the quantum (anti-)Zeno effect and the weak law of large numbers.

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1. Introduction

Consecutive observation of an unstable quantum system influences the decay behaviour of the system. If the decay is decelerated as a function of increasing observation frequency, this behaviour is referred to as the quantum Zeno effect. The limit of no decay at all for continuous observations, associated with survival probability one and paraphrased by the metaphor 'a watched pot never boils', was first pointed out by Misra and Sudarshan (1977). Since then, considerable progress has been made towards the theoretical understanding and experimental investigation of the effect (Itano *et al* (1990), Peres (1993), Namiki *et al* (1997), Facchi *et al* (2001), Roy (2001), Misra and Antoniou (2003), Gutiérrez-Medina *et al* (2003); see also Gustafson (2003) for some historical and other comments).

The quantum Zeno effect implies non-exponential contributions to the statistical decay law. In addition, it connects to the uncertain relationship between time and energy. More recently, the quantum anti-Zeno effect has been described according to which the decay of an unstable system is accelerated rather than decelerated (Kofman and Kurizki 2000). See Roy (2001) and Misra and Antoniou (2003) for reviews and Gutiérrez-Medina *et al* (2003) for recent experimental observations.

Consider an unstable quantum state $|\psi\rangle$ evolving under the Hamiltonian H. Criteria for the occurrence of the quantum Zeno effect can be given in terms of the survival amplitude $A(t) = \langle\psi|\,\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle$ of $|\psi\rangle$. According to the literature, the condition

$$\lim_{n\to\infty} |A(t/n)|^{2n} = 1 \qquad \text{for every} \quad t \in \mathbb{R} \tag{1}$$

provides a compact mathematical characterization of the quantum Zeno effect in terms of the limiting behaviour of the survival probability $|A|^2$ for an increasing number n of one-dimensional projection measurements within a given time interval. An expansion of both $|\psi\rangle$ and $|\,\mathrm{e}^{-\mathrm{i}tH}\psi\rangle$ in energy eigenstates shows that the survival amplitude can be expressed as the Fourier transform of the state energy (probability) density $|\lambda(E)|^2$,

$$A(t) = \int \mathrm{e}^{-\mathrm{i}tE}|\lambda(E)|^2\,\mathrm{d}E.$$

The problem is thus recast into a probabilistic framework, in which Luo *et al* (2002) formulated and established conditions for the quantum Zeno effect on the basis of general results for characteristic functions (i.e., Fourier transforms) and moments of probability distributions. They proved that (1) holds true if the first absolute moment of the state energy distribution is finite, i.e. if $\int_{-\infty}^{\infty} |E||\lambda(E)|^2\,\mathrm{d}E < \infty$, and they conjectured that this condition is also necessary.

In this paper, we show that the occurrence of the quantum Zeno effect actually depends on more subtle features of the state energy distribution, and that these are connected to the weak law of large numbers. The key observation leading to this conclusion is an alternative interpretation of the term $A(t/n)^n$ as the characteristic function of the mean value of n energy measurements for an ensemble of identical quantum systems prepared in the state $|\psi\rangle$. This interpretation differs from the usual one according to which $A(t/n)^n$ represents the probability amplitude of finding the state $|\psi\rangle$ undecayed in n consecutive energy measurements at times $t/n, 2t/n, \ldots, t$. Nevertheless, $A(t/n)^n$ is the crucial quantity for both situations.

In sections 2 and 3, we formulate and prove two theorems establishing necessary and sufficient conditions for the quantum Zeno and anti-Zeno effect, respectively. Section 4 contains a summary and discussion of the results.

2. Quantum Zeno effect

Our probabilistic framework involves independent random variables $X_1, X_2, \ldots$ distributed with the common law $\Pr(X_k < x) = F(x)$. Here F is a probability distribution function on the real line $\mathbb{R}$, i.e., a nondecreasing, left continuous function with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. Its characteristic function or Fourier transform is given by

$$\varphi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}\,\mathrm{d}F(x).$$

Applied to the quantum Zeno effect, F corresponds to the state energy distribution, the random variables X_k correspond to the outcomes of energy measurements of the individual systems forming the ensemble, and $\varphi(t)$ corresponds to the survival amplitude $A(t)$. The distribution function F may have discontinuities, thus covering the general case of a spectrum with both continuous and discrete components. Its decay at infinity will be characterized by the quantity

$$\delta_F(x) = x\Pr(|X_k| > x) = x(F(-x) + 1 - F(x)) \qquad x \geqslant 0$$

where the second equality holds at all points of continuity of F.

Theorem 1. *Under the above conditions the following three statements are equivalent:*

(a) $\lim_{n\to\infty} |\varphi(t/n)|^{2n} = 1$ *for every* $t \in \mathbb{R}$.
(b) $\lim_{x\to\infty} \delta_F(x) = 0$.
(c) For all $\epsilon > 0$ *we have*

$$\lim_{n\to\infty} \inf_{\mu\in\mathbb{R}} \Pr\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) = 0$$

i.e., there exists a sequence of real numbers μ_n *such that the distribution of the re-centred averages* $(X_1 + \cdots + X_n)/n - \mu_n$ *converges weakly to the Dirac measure at zero.*

Proof. The weak law of large numbers in probability theory, theorem 1 of Feller (1971, p 235), states the equivalence of (b) and (c). Hence, it suffices to prove the equivalence of (a) and (b). Clearly, condition (a) holds if and only if

$$\lim_{n\to\infty} (\gamma(t/n))^n = 1 \qquad \text{for every} \quad t \in \mathbb{R} \tag{2}$$

where $\gamma(t) = |\varphi(t)|^2$. Note that γ is the characteristic function of the difference $Y = X' - X''$ of independent random variables X', X'' each with distribution F. Let G denote the distribution function of Y. Then $\gamma(t/n)^n$ is the characteristic function of the mean $(Y_1 + \cdots + Y_n)/n$ of independent random variables $Y_1, Y_2, \ldots$ with common distribution G. By the continuity theorem for characteristic functions, (2) holds if and only if the averages $(Y_1 + \cdots + Y_n)/n$ converge to zero in distribution. Again by the weak law of large numbers, and since G is a symmetric distribution, condition (2) is equivalent to

$$\lim_{x\to\infty} \delta_G(x) = 0. \tag{3}$$

According to the symmetrization inequalities in lemma 1 of Feller (1971, p 149), there exists a real number a such that for all $x > 0$ one has

$$\frac{1}{2}\Pr(|X'| > x + a) \leqslant \Pr(|X' - X''| > x) \leqslant 2\Pr\left(|X'| > \frac{x}{2}\right). \tag{4}$$

It follows that (3) and (b) are equivalent, hence (a) and (b) are equivalent as well. The proof of theorem 1 is complete. □

If the first absolute moment $M_1(F) = \int_{-\infty}^{\infty} |x| \, \mathrm{d}F(x)$ of F is finite then

$$\begin{aligned}
\lim_{x\to\infty} \delta_F(x) &= \lim_{x\to\infty} \left(\int_{-\infty}^{-x} y \, \mathrm{d}F(y) + \int_{x}^{\infty} y \, \mathrm{d}F(y)\right) \\
&\leqslant \lim_{x\to\infty} \left(\int_{-\infty}^{-x} |y| \, \mathrm{d}F(y) + \int_{x}^{\infty} |y| \, \mathrm{d}F(y)\right) \\
&= M_1(F) - \lim_{x\to\infty} \int_{-x}^{x} |y| \, \mathrm{d}F(y) = 0
\end{aligned}$$

so that condition (b) holds. However, there exist distribution functions F for which $M_1(F)$ is infinite whereas (b) remains valid, such as the distribution function defined by

$$F(x) = 1 - \frac{e}{x \ln x} \qquad x \geqslant e$$

and $F(x) = 0$ for $x \leqslant e$. This provides an explicit counterexample to the aforementioned conjecture of Luo *et al* (2002), showing that the quantum Zeno effect can occur even if the state energy distribution does not have a finite first moment. For large x, corresponding to high energies E, the density function $F'(x)$ in the above example differs from the Breit–Wigner distribution with the density function $\pi^{-1}(1 + x^2)^{-1}$ by a logarithmic factor. Luo *et al* (2002) treated the Breit–Wigner distribution as the special case $\nu = 1/2$ of a parametric family of probability densities of the form $f_\nu(x) = c(\nu)(1 + x^2)^{-\nu-1/2}$ and noted that it corresponds to a critical case separating the quantum Zeno and anti-Zeno effect within this family. Indeed, for $\nu > 1/2$ condition (b), hence (a) holds, whereas for $\nu < 1/2$ one has

$$\lim_{n\to\infty} |\varphi(t/n)|^{2n} = 0 \qquad \text{for} \quad t \neq 0 \tag{5}$$

where $\varphi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} f_\nu(x)\,\mathrm{d}x$ denotes the characteristic function of the density f_ν. The criterion (5) is dual to (a) and is used here, following Luo *et al* (2002), to characterize the quantum anti-Zeno effect. This qualitative definition of the anti-Zeno effect refers to the short-time behaviour of the survival amplitude and differs from other formulations in the literature; related comments may be found in section 4.

3. Quantum anti-Zeno effect

In theorem 2 we show that the quantity $\delta_F(x) = x(F(-x) + 1 - F(x))$ not only characterizes the decay behaviour of the state energy distribution required for the quantum Zeno effect as in theorem 1, but plays an analogous role for the quantum anti-Zeno effect (as characterized by (5)), apart from a mild regularity condition. We say that F is a *straight* distribution function if either $\sup_{x>0} \delta_F(x) < \infty$ or $\lim_{x\to\infty} \delta_F(x) = \infty$. This excludes the case where $\limsup_{x\to\infty} \delta_F(x) = \infty$ while $\lim_{x\to\infty} \delta_F(x)$ does not exist. Roughly speaking, straightness of F requires that the energy spectrum has no sequence of gaps of increasing lengths.

Theorem 2. *For straight distribution functions F the following three conditions are equivalent:*

(d) $\lim_{n\to\infty} |\varphi(t/n)|^{2n} = 0$ *in measure, i.e., for all* $T > 0$ *and* $\epsilon > 0$ *the Lebesgue measure of the set of all* $|t| < T$ *with* $|\varphi(t/n)|^{2n} > \epsilon$ *converges to zero.*

(e) $\lim_{x\to\infty} \delta_F(x) = \infty$.

(f) For all $c > 0$ *we have*

$$\lim_{n\to\infty} \sup_{\mu\in\mathbb{R}} \Pr\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \leqslant c\right) = 0$$

i.e., the distribution of the average $(X_1 + \cdots + X_n)/n$ *becomes entirely spread out as* $n \to \infty$.

Proof. The *concentration function* of a distribution function H is defined as the maximal H-probability of an interval of length ℓ,

$$Q_H(\ell) = \sup_{x\in\mathbb{R}} (H(x + \ell/2) - H(x - \ell/2)).$$

Let η be the characteristic function of H. The main lemma of Esseen (1968) states that there exist constants C_1 and C_2 such that for every $b > 0$ and every a satisfying $0 < a \leqslant \pi/\ell$, one has

$$C_1 \frac{\ell}{1 + 2b\ell} \int_{-b}^{b} |\eta(t)|^2\,\mathrm{d}t \leqslant Q_H(\ell) \leqslant C_2 a^{-1} \int_{-a}^{a} |\eta(t)|\,\mathrm{d}t. \tag{6}$$

The equivalence of conditions (d) and (f) is an immediate consequence of these inequalities. Indeed, put $H = H_n$ where $H_n(x) = \Pr((X_1 + \cdots + X_n)/n \leqslant x)$ is the distribution function of the random variable $(X_1 + \cdots + X_n)/n$. Then $\eta(t)$ is the characteristic function $\varphi(t/n)^n$ of H_n. Since characteristic functions are uniformly bounded in absolute value, the two integrals in (6) tend to zero if and only if (d) holds. The equivalence of (d) and (f) then is immediate upon noting that

$$\sup_{\mu\in\mathbb{R}} \Pr\left(\left|\frac{X_1+\cdots+X_n}{n}-\mu\right|\leqslant c\right) = Q_{H_n}(2c).$$

The proof that (e) implies (d) again uses symmetrization. Let

$$\Delta_G(x) = \int_{|y|\leqslant x} (y/x)^2 \,\mathrm{d}G(y) + \int_{|y|>x} \mathrm{d}G(y)$$

for $x > 0$, where G is defined as in the proof of theorem 1. An application of theorem 3.1 of Esseen (1968) shows that there exists a constant C such that for every $\ell > 0$

$$Q_{H_n}(\ell) \leqslant C(n\Delta_G(n\ell))^{-1/2} \tag{7}$$

with H_n as defined above. Putting $\ell = 1$ in (6) and (7) and using the trivial estimate $n\Delta_G(n) \geqslant \delta_G(n)$ gives

$$\frac{C_1}{1+2b}\int_{-b}^{b} |\varphi(t/n)|^{2n}\,\mathrm{d}t \leqslant Q_{H_n}(1) \leqslant C\delta_G(n)^{-1/2}$$

for every $b > 0$. By the symmetrization inequalities (4), condition (e) is equivalent to $\lim_{x\to\infty}\delta_G(x) = \infty$. Hence, the chain of inequalities shows that (e) implies (d).

It remains to prove that, conversely, (d) implies (e). Putting again $\gamma = |\varphi|^2$ and writing

$$|\varphi(t/n)|^{2n} = \left(1 - \frac{n(1-\gamma(t/n))}{n}\right)^n$$

shows that $n(1-\gamma(t/n))$ becomes large if and only if the left-hand side becomes small. Therefore, (d) implies that $\exp(-n(1-\gamma(t/n)))$ converges to zero in measure. Jensen's inequality (Feller 1971, p 153) then gives

$$\lim_{n\to\infty} \exp\left(-\int_0^1 n(1-\gamma(t/n))\,\mathrm{d}t\right) \leqslant \lim_{n\to\infty}\int_0^1 \exp(-n(1-\gamma(t/n)))\,\mathrm{d}t = 0$$

so that

$$\lim_{n\to\infty}\int_0^1 n(1-\gamma(t/n))\,\mathrm{d}t = \infty.$$

Another classical inequality (Loève 1960, pp 195–6) applied to the characteristic function $\gamma(t/n)$ of the distribution function $G_n(x) = G(nx)$ shows that there exists a constant $C_0 > 0$ such that

$$C_0\int_0^1 (1-\gamma(t/n))\,\mathrm{d}t \leqslant \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,\mathrm{d}G_n(x) \leqslant \Delta_{G_n}(1) = \Delta_G(n).$$

Consequently, $\lim_{n\to\infty} n\Delta_G(n) = \infty$. Suppose then that $\limsup_{n\to\infty}\delta_G(n) < \infty$. Integration by parts applied to the first term of $n\Delta_G(n)$ gives

$$n\int_{|x|\leqslant n} (x/n)^2\,\mathrm{d}G(x) = \frac{1}{n}\int_0^n 2\delta_G(x)\,\mathrm{d}x - \delta_G(n)$$

which implies that $\limsup_{n\to\infty} n\Delta_G(n) < \infty$. The contradiction shows that we have $\limsup_{n\to\infty}\delta_G(n) = \infty$, hence $\limsup_{n\to\infty}\delta_F(n) = \infty$ by (4), and since F is straight (e) holds. The proof of theorem 2 is complete. □

4. Summary and discussion

Refining recent work by Luo *et al* (2002), we formulate and prove two theorems providing necessary and sufficient conditions for the quantum Zeno and anti-Zeno effect in a unified manner, by connecting these effects to the weak law of large numbers. The conditions refer to the survival probability of an unstable quantum state, its state energy distribution, and the distribution of the ensemble average of related energy measurements.

The relation between the quantum (anti-)Zeno effect and the weak law of large numbers draws on the fact that the survival amplitude of an unstable quantum state during n repeated measurements at successive times $t/n, 2t/n, \ldots, t$ may also be interpreted as the characteristic function of an ensemble average. In physical terms, the two theorems express examples for the complementarity of time and energy and refer to corresponding time–energy uncertainty relations: a large survival probability, corresponding to a long lifetime of an unstable state, is related to a narrow energy distribution in the case of the Zeno effect, and vice versa in the case of the anti-Zeno effect.

The critical situation at the transition between Zeno and anti-Zeno behaviour corresponds to a state energy distribution F for which the probability of measuring an energy larger than E decays roughly such as E^{-1}. This situation also describes a critical region with respect to the weak law of large numbers. Depending on the precise decay behaviour of F, the distribution of the ensemble average of energy measurements may either shrink to a single value (Zeno effect), or spread out entirely (anti-Zeno effect), or stay within a bounded region essentially as the number n of measurements becomes large.

In the literature (e.g., Facchi *et al* 2001), the quantum (anti-)Zeno effect is often defined in terms of deviations from a characteristic exponential decay behaviour of the function $\tau \mapsto P(\tau)^n = |A(\tau)|^{2n}$. This function represents the survival probability of an unstable quantum system at time $t = n\tau$ if n measurements are made at equal temporal distances τ within the interval $[0, t]$. According to Fermi's 'golden' rule, the overall decay of $P(\tau)^n$ as a function of total measurement time $t = n\tau$ goes roughly as $\mathrm{e}^{-\gamma_0 t}$, with γ_0^{-1} the 'natural' lifetime. However, there may be an initial period where $P(\tau)^n$ exceeds $\mathrm{e}^{-\gamma_0 n\tau}$ (for instance if $P(\tau)$ has a quadratic short-time behaviour), followed by an intermediate period where $P(\tau)^n$ falls below $\mathrm{e}^{-\gamma_0 n\tau}$. In the first case, the effective decay rate is smaller than γ_0, giving rise to the quantum Zeno effect, while in the second case it exceeds γ_0, so the decay is accelerated for such values of τ and the anti-Zeno effect occurs. The exact exponential decay as in Fermi's rule, $P_0(\tau)^n = \mathrm{e}^{-\gamma_0 n\tau}$, implies an invariance property with respect to n, namely

$$P_0(t/n)^n = \mathrm{e}^{-\gamma_0 nt/n} = \mathrm{e}^{-\gamma_0 t} = P_0(t)$$

independently of n. Thus the re-scaling $\tau = t/n$ applied in our approach focuses on those times τ between measurements for which $P_0(\tau)^n$ shows a non-trivial behaviour as n tends to infinity, permitting proper distinction between decelerated and accelerated decay. Deviations from an exponential decay are scaled up by the asymptotics $n \to \infty$, and in the limit the survival probabilities reduce to 1 and 0, respectively. Our theorems give the exact conditions for either of the two alternatives.

Basically, in Facchi *et al* (2001) and elsewhere, the number n of measurements is fixed, and the time τ between measurements and the total measurement period $t = n\tau$ are varied. In the present paper t is fixed, and n and $\tau = t/n$ are varied. In particular, τ becomes small as $n \to \infty$ and the short-time evolution of $P(\tau)$ (or else, the precise decay of the energy distribution) is decisive for the distinction between Zeno and anti-Zeno behaviour within the bounded measurement interval $[0, t]$. Our approach thus does not cover anti-Zeno behaviour at later times, preceded by an initial Zeno period.

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